Analytical solution of wave system in $\Re^n$ with coupling controllers

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Abstract: In this paper, the homotopy perturbation method (HPM) is employed to find solution of wave system in $\Re^n$ with coupling controllers. The analytical solution is calculated in the form of convergence power series with easily computable components. The homotopy perturbation method performs extremely well in terms of accuracy, efficiently, simplicity, stability and reliability.

Keywords: Padé approximation, Homotopy perturbation method, Wave equations.

1. Introduction

This paper considers analytical solution of wave system in $\Re^n$ with coupling controllers by using the homotopy perturbation method. The governing equation for the distributed parameter control problem can be modeled as follows:

$$u_{tt} - c_1^2 \Delta u = l(v-u) + \beta(v_t - u_t), \qquad \text{in } \Omega_u \times (0,\infty), \tag{1}$$

$$u_{tt} - c_2^2 \Delta v = l(u-v) + \beta(u_t - v_t), \qquad \text{in } \Omega_v \times (0,\infty), \tag{2}$$

with initial conditions

$$u(0) = f_1, \qquad u_t(0) = g_1, \qquad \text{in } \Omega_u, \tag{3}$$

$$v(0) = f_2, \qquad v_t(0) = g_2, \qquad \text{in } \Omega_v, \tag{4}$$

Along with Eqs.(1)-(2), we employ the following homogeneous Dirichlet boundary condition as

$$u = 0, \quad v = 0, \quad \text{on } \partial\Omega \times [0,\infty). \tag{5}$$

Recently, Najafi [1] used Adomian decomposition method for solving the governing equation. In this paper, we will use HPM for this problem. The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [2-5]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of ''deformations'', the solution for each of which is ''close'' to that at the previous stage of ''deformation''. Eventually at $p=1$, the system takes the original form of the equation and the final stage of ''deformation'' gives the desired solution. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution. This technique has been employed to solve a large variety of linear and nonlinear problems [6-22, 39-44]. The interested reader can see the Refs. [23-26, 31-36, 39-44] for last development of HPM.

2. Application of HPM

We would like to apply HPM to the system of linear partial differential equations, i.e., the system of waves in $\Re^2$, the two-dimensional version of system (1)-(2) in Section 1. This problem is motivated by an analogous problem in ordinary differential equations for coupled oscillators and has potential application in isolating a vibrating object from the outside disturbances. For example, rubber or rubber-like materials can be used to either absorb or shield a structure from vibration. As an approximation, these materials can be modeled as distributed springs. For further applications of such a configuration, interested readers are referred to [27-29, 31-36].

The dynamics of the system under consideration are governed by the following set of partial differential equations for coupled wave equations in $\Omega \subset \Re^2$ and boundary $\partial\Omega$:

$$u_{tt} = c_1^2 \Delta u + l(v-u) + \beta(v_t - u_t),$$

in $\Omega \times (0, \infty)$, (6)

$$v_{tt} = c_2^2 \Delta v + l(u - v) + \beta(u_t - v_t),$$

with initial conditions,

$$u(0) = f_1 = \sin(\pi x)\sin(\pi y), \qquad v(0) = f_2 = \sin(\pi x)\sin(\pi y),$$

$$u_t(0) = g_1 = 0, \qquad v_t(0) = g_2 = 0, \qquad \text{in } \Omega \tag{7}$$

and Dirichlet boundary conditions,

$$u = v = 0, \qquad \text{on } \partial\Omega \times [0, \infty). \tag{8}$$

To solve equation by the homotopy perturbation method, we construct the following homotopy

$$u_{tt} = p[c_1^2 \Delta u + l(v - u) + \beta(v_t - u_t)], \tag{9}$$

$$v_{tt} = p[c_2^2 \Delta v + l(u - v) + \beta(u_t - v_t)], \tag{10}$$

Assume the solution of Eqs. (9)-(10) in the forms:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots \tag{11}$$

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \tag{12}$$

Substituting Eqs.(11)-(12) into Eqs. (9)-(10) and collecting terms of the same power of p give

$$p^0 : (u_0)_{tt} = 0, \tag{13}$$

$$p^1 : (u_1)_{tt} = c_1^2 \Delta u_0 + l(v_0 - u_0) + \beta((v_0)_t - (u_0)_t), \tag{14}$$

$$p^2 : (u_2)_{tt} = c_1^2 \Delta u_1 + l(v_1 - u_1) + \beta((v_1)_t - (u_1)_t), \tag{15}$$

$$p^3 : (u_3)_{tt} = c_1^2 \Delta u_2 + l(v_2 - u_2) + \beta((v_2)_t - (u_2)_t), \tag{16}$$

$$\dots$$

and

$$p^0 : (v_0)_{tt} = 0, \tag{17}$$

$$p^1 : (v_1)_{tt} = c_2^2 \Delta v_0 + l(v_0 - u_0) + \beta((u_0)_t - (v_0)_t), \tag{18}$$

$$p^2 : (v_2)_{tt} = c_2^2 \Delta v_1 + l(v_1 - u_1) + \beta((u_1)_t - (v_1)_t), \tag{19}$$

$$p^3 : (v_3)_{tt} = c_2^2 \Delta v_2 + l(v_2 - u_2) + \beta((u_2)_t - (v_2)_t), \tag{20}$$

$$\vdots$$

If we solve the above system of equations, we successively obtain

$$u_0 = \sin(\pi x)\sin(\pi y), \tag{21}$$

$$u_1 = -c_1^2 \pi^2 \sin(\pi x)\sin(\pi y) t^2, \tag{22}$$

$$u_2 = \left[\frac{\beta \pi^2}{3}(c_1^2 - c_2^2)t^3 + \frac{1}{6}\left(c_1^4 \pi^4 - \frac{\pi^2}{2}c_2^2 + \frac{\pi^2}{2}c_1^2 \right) t^4 \right] \sin(\pi x)\sin(\pi y), \tag{23}$$

$$\dots$$

$$v_0 = \sin(\pi x)\sin(\pi y), \tag{24}$$

$$v_1 = -c_2^2 \pi^2 \sin(\pi x)\sin(\pi y) t^2, \tag{25}$$

$$v_2 = \left[\frac{\beta \pi^2}{3}(c_2^2 - c_1^2)t^3 + \frac{1}{6}\left(c_2^4 \pi^4 - \frac{\pi^2}{2}c_1^2 + \frac{\pi^2}{2}c_2^2 \right) t^4 \right] \sin(\pi x)\sin(\pi y), \tag{26}$$

$$\dots$$

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Then the series solutions expression by HPM can be written in the form:

$$u = u_0 + u_1 + u_2 + u_3 + \dots \tag{27}$$

$$v = v_0 + v_1 + v_2 + v_3 + \dots \tag{28}$$

So we obtain the series solutions

$$u(x,t)=\left[1-c_1^2\pi^2t^2+\left(\frac{1}{3}\beta\pi^2c_1^2-\frac{1}{3}\beta\pi^2c_2^2\right)t^3+\left(\frac{1}{6}\pi^4c_1^4-\frac{1}{12}\pi^2c_2^2+\frac{1}{12}\pi^2c_1^2\right.\right.$$

$$-\frac{1}{6}\beta^2\pi^2c_1^2+\frac{1}{6}\beta^2\pi^2c_2^2\Bigg)t^4+\left(-\frac{\beta\pi^4c_1^4}{15}+\frac{\beta\pi^4c_1^2c_2^2}{30}+\frac{\beta\pi^2c_2^2}{15}-\frac{\beta\pi^2c_1^2}{15}+\frac{\beta\pi^4c_2^4}{30}\right)t^5 \qquad (29)$$

$$\left.+\left(\frac{\pi^2c_2^4}{180}-\frac{\pi^2c_1^2}{180}+\frac{\pi^2c_2^2}{180}-\frac{\pi^4c_1^4}{90}-\frac{\pi^6c_1^6}{90}+\frac{\pi^4c_1^2c_2^2}{180}\right)t^6\right]\sin(\pi x)\sin(\pi y).$$

Similarly,

$$v(x,t)=\left[1-c_{21}^2\pi^2t^2+\left(-\frac{1}{3}\beta\pi^2c_1^2+\frac{1}{3}\beta\pi^2c_2^2\right)t^3+\left(\frac{1}{12}\pi^2c_2^4-\frac{1}{12}\pi^2c_1^2+\frac{1}{6}\beta^2\pi^2c_1^2\right.\right.$$

$$-\frac{1}{6}\beta^2\pi^2c_2^2+\frac{1}{6}\pi^4c_2^4\Bigg)t^4+\left(\frac{\beta\pi^4c_1^4}{30}+\frac{\beta\pi^4c_1^2c_2^2}{30}-\frac{\beta\pi^2c_2^2}{15}+\frac{\beta\pi^2c_1^2}{15}-\frac{\beta\pi^4c_2^4}{315}\right)t^5 \qquad (30)$$

$$\left.+\left(-\frac{\pi^4c_2^4}{90}+\frac{\pi^2c_1^2}{180}-\frac{\pi^2c_2^2}{180}+\frac{\pi^4c_1^4}{90}-\frac{\pi^6c_{21}^6}{90}+\frac{\pi^4c_1^2c_2^2}{180}\right)t^6\right]\sin(\pi x)\sin(\pi y).$$

Now, in order to find a closed form solution for this system in (6), we apply the Aftertreatment Technique (AT) by Jiao in [30] .To do this, applying the Laplace transform to the coefficients of $\sin(\pi x)\sin(\pi y)$ in (29) yields

$$l(u(x,t))=\{1/s^7\,[(-8c_1^4\pi^4-4c_1^2\pi^2+4c_2^2\pi^2+4c_1^2c_2^2\pi^4+4c_2^4\pi^4-8c_1^6\pi^6)$$

$$+(-8\beta c_1^2\pi^2+4c_2^4\pi^4+8\beta c_2^2\pi^2+4\beta c_1^2c_2^2\pi^4-8c_1^4\pi^4)s$$

$$+(4\beta^2c_2^2\pi^2-4\beta^2c_1^2\pi^2-2c_2^2\pi^2+4c_1^4\pi^4)s^2+(-2\beta c_2^2\pi^2+2\beta c_1^2\pi^2)s^3 \qquad (31)$$

$$-2c_1^2\pi^2s^4+s^6]\sin(\pi x)\sin(\pi y)\}.$$

For the sake of simplicity we let $s=1/\xi$; then (31) becomes

$$
\begin{aligned}
\bar{l}(u(x,t)) = \Big\{ \Big[& \left(-8c_1^4\pi^4 - 4c_1^2\pi^2 + 4c_2^2\pi^2 + 4c_1^2c_2^2\pi^4 + 4c_2^4\pi^4 - 8c_1^6\pi^6\right)\xi^7 \\
& + \left(-8\beta c_1^2\pi^2 + 4c_2^4\pi^4 + 8\beta c_2^2\pi^2 + 4\beta c_1^2 c_2^2\pi^4 - 8c_1^4\pi^4\right)\xi^6 \\
& + \left(4\beta^2 c_2^2\pi^2 - 4\beta^2 c_1^2\pi^2 - 2c_2^2\pi^2 + 4c_1^4\pi^4\right)\xi^5 \\
& + \left(-2\beta c_2^2\pi^2 + 2\beta c_1^2\pi^2\right)\xi^4 - 2c_1^2\pi^2\xi^3 + \xi \Big] \sin(\pi x)\sin(\pi y) \Big\}.
\end{aligned} \tag{32}
$$

The use of Pade´ approximants shows real promise in solving boundary value problems in an infinite domain; see [24-26, 37, 38]. It is well known in the literature that polynomials are used to approximate the truncated power series and tend to exhibit oscillations that may give an approximation error bounds. Moreover, polynomials can never blow up in a finite plane and this makes the singularities not apparent. To overcome these difficulties, the obtained series is best manipulated by Pade´ approximants for numerical approximations. Using the power series, isolated from other concepts, is not always useful because the radius of convergence of the series may not contain the two boundaries. It is now well known that Pade´ approximants have the advantage of manipulating the polynomial approximation into rational functions of polynomials. By this manipulation, we gain more information about the mathematical behavior of the solution. In addition, the power series are not useful for large values of x. It is an established fact that power series in isolation are not useful to handle boundary value problems. This can be attributed to the possibility that the radius of convergence may not be sufficiently large to contain the boundaries of the domain. It is therefore essential to combine the series solution with the Pade´ approximants to provide an effective tool to handle boundary value problems on an infinite or semi-infinite domain. Utilizing the Pade´ approximation $\left[\frac{2}{2}\right]$, to approximate (32) yields

$$
\left[\frac{2}{2}\right]_{\xi} = \frac{c_1^2\xi + \left(\beta c_1^2 - \beta c_2^2\right)\xi^2}{c_1^2 + \left(\beta c_1^2 - \beta c_2^2\right)\xi + 2c_1^4\pi^2\xi^2}. \tag{33}
$$

Let $\xi = 1/s$, then (33) becomes

$$
\left[\frac{2}{2}\right]_{s} = \frac{s+d}{s^2 + ds + 2\pi^2 c_1^2}, \qquad \text{where } d = \frac{\left(\beta c_1^2 - \beta c_2^2\right)}{c_1^2}. \tag{34}
$$

Finally, applying the inverse Laplace transform to (34) results in the following analytical approximate solution in the closed form for (6) in $\Re^2$:

$$u = e^{\left(-\frac{d}{2}\right)t}\left[\cosh\left(\frac{1}{2}\sqrt{d^2-8c_1^2\pi^2}\right)t + + \frac{d\sinh\left(\frac{1}{2}\sqrt{d^2-8c_1^2\pi^2}\right)t}{\left(\sqrt{d^2-8c_1^2\pi^2}\right)}\right]\sin(\pi x)\sin(\pi y). \quad (35)$$

Similarly,

$$v = e^{\left(-\frac{d}{2}\right)t}\left[\cosh\left(\frac{1}{2}\sqrt{d^2-8c_2^2\pi^2}\right)t + + \frac{d\sinh\left(\frac{1}{2}\sqrt{d^2-8c_2^2\pi^2}\right)t}{\left(\sqrt{d^2-8c_2^2\pi^2}\right)}\right]\sin(\pi x)\sin(\pi y). \quad (36)$$

3. Conclusion

In this paper, we used HPM to obtain analytical solution of wave system in $\Re^n$ with coupling controllers. The method provides the solutions in the form of a series with easily computable terms. Unlike other common methods for solving physical problem, linear or nonlinear, that requires linearization, discretization, perturbation, or unjustified assumptions, that may slightly change the physics of the problem, the HPM finds approximate analytical solutions by using the initial conditions only.

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